A Probabilistic Analysis of a Fixed Partition Policy for the Inventory-Routing Problem

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Abstract: We consider the Inventory-Routing Problem (IRP) where n geographically dispersed retailers must be supplied by a central facility. The retailers experience demand for the product at a deterministic rate, and incur holding costs for keeping inventory. Distribution is performed by a fleet of capacitated vehicles. The objective is to minimize the average transportation and inventory costs per unit time over the infinite horizon. We focus on the set of Fixed Partition Policies (FPP). In an FPP, the retailers are partitioned into disjoint and collectively exhaustive sets. Each set of retailers is served independently of the others and at its optimal replenishment rate. Previous research has measured the effectiveness of an FPP solution relative to a lower bound over all policies. We propose an additional measure that is relative to the optimal FPP. In this paper we construct a polynomial-time partitioning scheme that is shown to yield an FPP whose cost is asymptotically within $1.5\% + \epsilon$ of the cost of an optimal FPP, for arbitrary $\epsilon > 0$. In addition, in some cases, our polynomial-time scheme yields an FPP whose cost is asymptotically within $1.5\% + \epsilon$ of the minimal policy's cost (over all feasible policies). © 2004 Wiley Periodicals, Inc. Naval Research Logistics 51: 925–948, 2004.

1. INTRODUCTION

In many distribution systems significant cost reductions and service improvements may be achieved by adopting efficient inventory replenishment strategies for all facilities concerned. Efficient strategies usually exploit economies of scale, i.e., by shipping full (or close to full) vehicles that combine deliveries to distinct locations into efficient routes. The derivation of such strategies requires a careful coordination of various logistical planning functions, particularly the areas of transportation planning and inventory control.

Recently, the popularity of approaches going under the heading of *vendor managed inventory* (VMI) has increased. According to this approach the responsibility for replenishment decisions

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is shifted from the buyer to the suppliers. Among other benefits, this method allows the supplier to reduce the transportation and inventory cost by a careful design of the routing system. A successful implementation of the VMI approach requires access to accurate and updated information about the stock levels at the retailers. In spite of its popularity, the VMI is usually applied only to small-to-medium systems. Large-scale systems, such as the ones analyzed in this paper, are too complex for a successful coordination of the various activities by VMI.

The ability to reduce costs as well as to improve service shows that an appropriately managed logistics system can be the key to enhancing the company's competitive edge. Stalk, Evans, and Shulman [25] attribute Wal-Mart's impressive success to its replenishment strategy. Wal-Mart logistics is based on "cross-docking" points (e.g., warehouses or retailers) that are responsible for taking in shipments from vendors and delivering them to a lower echelon (e.g., retailers or customers). The sole function of cross-docking points is to coordinate among the various activities in the supply process; they do not hold stock by themselves. In this paper we consider such a two-echelon subsystem in which a single supplier is responsible for the replenishment of a set of retailers. More specifically, we consider an infinite-horizon Inventory-Routing Problem (IRP) where retailers face constant deterministic retailer-specific demand rates. The objective is to determine long-term *integrated* replenishment strategies (i.e., inventory rules and routing patterns) allowing all retailers to meet their demands while minimizing long-run average system-wide transportation and inventory costs.

It is generally perceived that an optimal policy for this problem may be quite complex and difficult to implement in practice. This has prompted research on this problem to concentrate on a specific policy class called *partitioning policies.* These are characterized by a given set of routes with the following properties: Each route is driven at equidistant time intervals (which are route-dependent) visiting a subset of the retailers. In addition, each route is responsible for replenishing a certain retailer-dependent fraction of the demand of each of its retailers. The literature distinguishes between two versions of the partitioning problem: the *split* and the *unsplit* demand case (see [2]). In the split demand case, a retailer may be served by a number of routes, whereas in the unsplit demand case, each retailer must be served on a single route. That is, in the unsplit demand case, a route either supplies all the demand of a given retailer or it doesn't visit it at all. A partitioning policy for the unsplit demand case is called a *fixed partitioning policy* (FPP). Clearly, an FPP is also a feasible solution for the split demand case but not vice versa. Even though both versions of the problem are NP-hard, the split demand case is usually perceived as simpler since it allows more flexibility in combining deliveries on a route so that the vehicles' capacity is better exploited. The unsplit demand case has the additional complexity of requiring the retailers to be partitioned into *disjoint* sets.

The partition of the retailers into sets can be performed in various ways. For example, for the unsplit demand case, Bramel and Simchi-Levi [8] partition the retailers by solving a Capacitated Concentrator Location Problem (CCLP) which is formulated as an integer linear program. In the split demand case, it has been shown that simple polynomial partitioning methods, called *Region Partitioning Policies* (RPP), that divide the plane into small regions may be extremely effective. Anily and Federgruen in [1, 4, 5] construct RPPs that are asymptotically optimal within the class of partitioning policies. In [7] the authors consider two-echelon systems and design an RPP whose cost is asymptotically no more than 2% above the minimal policy cost. Unfortunately, applying an RPP scheme in the unsplit demand case may result in an inefficient solution that consists of an unnecessary large number of routes each serving a very small sales volume. In this paper we show how to overcome this difficulty by first implementing a region-partitioning scheme that partitions the retailers into disjoint regions, and then, repartitioning each such region by using a bin-packing heuristic to efficiently combine retailers into feasible routes.

The impetus for this line of research is the paper by Chan, Federgruen, and Simchi-Levi [9], who considered exactly the same model as the one considered here. These authors were the first to challenge the issue of analyzing the asymptotic effectiveness of the FPP class. The authors proved that the effectiveness of the optimal FPP is closely related to the *packing constant* in the associated bin packing problem which assumes values in the range [1, 2]: The value 1 is associated with *perfect packing,* i.e., cases where asymptotically the bin (vehicle) capacities are fully utilized by an optimal solution. On the other hand, bin-packing problems whose packing constant is 2 are the "worst" since asymptotically only 50% of the bins' capacity is utilized by an optimal solution. Chan, Federgruen, and Simchi-Levi [9] proved that the asymptotic ratio of the cost of an optimal FPP to a lower bound for the optimal cost value under *any* strategy is no more than the square root of the packing constant of the associated bin-packing problem. Thus, for problems that allow perfect packing, the cost of an optimal FPP converges asymptotically to the optimal cost over all strategies. But if the packing constant is 2, then the cost of an optimal FPP may be asymptotically as $\sqrt{2}$ ($\approx$ 141%) times the optimal cost over all strategies. Yuyue [26] used the same methodology to investigate the same problem as in Chan, Federgruen, and Simchi-Levi [9] with a more general cost structure, and proved that the upper bound is asymptotically at most 1.5 times the lower bound. However, in problems where the induced bin-packing problem allows for perfect packing, the asymptotic effectiveness measure is reduced to 1.06.

In order to better understand the effect of the various restrictions on the policy structure, we developed an alternative effectiveness measure of an FPP which is relative to the optimal FPP cost. More importantly, this measure enables us to design an FPP whose cost is asymptotically very close to the optimal FPP cost. Both measures of effectiveness, i.e., with respect to a lower bound over all policies, or relative to the cost of the optimal FPP, are informative in making decisions regarding the desirable structure of the policy that will be applied. We note however that focusing *a priori* on the FPP class may be costly. For example, if the cost of the FPP in use exceeds the lower bound on the optimal cost under any strategy by 30% but it exceeds the cost of the optimal FPP by 1%, then the user may want to reconsider the restriction to the FPP class and see whether it can be relaxed in order to reduce the costs. However, if the two effectiveness measures are close then the user may feel comfortable with the restriction to the FPP class in view of its simplicity as is explained below.

Unfortunately, none of the above mentioned research on FPPs provide the user with a polynomial-time algorithm to determine a policy in the FPP class with an *ex ante* bound on its optimality gap (within the FPP class). The only exception is the result on direct shipping of [15]. The main reason for the relatively slow progress in this direction is due to the lack of a good lower bound on the optimal cost in the FPP class. In [3], the authors develop a lower bound on the optimal cost *within the FPP class,* which is shown to be asymptotically at least 98.5% of the optimal cost.

The restriction to the FPP class arises in many real applications mainly in order to simplify the routing pattern of the various trucks and to eliminate the need for coordination of the routes with respect to the time they are driven. Policies that split the service to a retailer to different routes suffer from a number of drawbacks. First, the extra setup time involved in stopping and unloading the merchandise at the retailer may be expensive. In addition, a retailer that is served on distinct routes may require a coordination of the timing of the visits by the different trucks in order to smooth the replenishments of its stock. A coordination of the tours that yields a replenishment pattern in which the retailers are served in equi-distant intervals of time may lower significantly the total holding cost of the system, see discussion in [6, 17]. However, such a coordination of the routes is computationally a very hard task. Another disadvantage of optimal policies for the split demand case lies in the fact that in such policies most of the vehicles leave the depot filled up to their capacity; as a result, the schedule is very sensitive to small variations in the retailers' demand rates. An FPP, on the other hand, usually uses trucks that are not fully loaded and therefore can more easily accommodate variations in the demand rates. For all these reasons, in many applications of IRP practitioners focus *a priori* on fixed partition policies.

In this paper we present a procedure which is based on a region partitioning scheme (RP) called the Circular Fixed Partitioning scheme (CFP) and we investigate its effectiveness relative to the lower bound on the FPP class that was derived in [3]. The scheme works as follows. In the first step, using region partitioning, we partition the plane into disjoint *clusters* of retailers that are in close proximity. In the second step, the retailers within a cluster are partitioned into a number of subsets by applying a bin-packing heuristic. Each of the resulting subsets is then served by a single vehicle using a periodic schedule. In addition, we prove that for a given ϵ, if the asymptotically ϵ-optimal bin-packing heuristic proposed in [14] is used, then the CFP scheme proposed here is polynomial and, moreover, its cost comes asymptotically within 1.5% + ϵ of the cost of an optimal FPP independently of the packing features of the problem!

In the next section we describe the probabilistic model and give some notation. In Section 3, we present CFP. In Section 4, we investigate the effectiveness of CFP and its relation to the effectiveness of the bin-packing heuristic used. In Section 5 we show that using the Set-Covering based bin-packing heuristic (see [19, 14]) in CFP results in an FPP whose cost is asymptotically at most 1.5% + ϵ above the cost of the best FPP.

2. PRELIMINARIES

Let $N = \{1, 2, \ldots, n\}$ denote the set of retailers. We denote by w_i the demand rate of retailer $i \in N$. For any $S \subseteq N$, let $w(S) \stackrel{\text{def}}{=} \sum_{i \in S} w_i$. In general, W will denote the sum of a set of retailer demand rates, while w will denote a particular retailer demand rate. An unlimited number of vehicles of capacity Q deliver the product to the retailers. The delivery cost consists of a fixed cost $c \geq 0$ plus a term that is proportional to the distance traveled. We assume the cost per mile is 1. We also assume a *frequency* constraint, i.e., a retailer may not be replenished at a rate greater than f. The holding cost is h per unit per unit of time, independent of the location.

An FPP is specified by a partition of N into a collection of exhaustive, disjoint nonempty sets $\{S_1, S_2, \ldots, S_m\}$. Note that a set $S \subseteq N$ can be *feasibly* served in an FPP if $w(S) \leq Qf$. If $w(S) > Qf$, then the set S cannot be served on a single route since even if the vehicle were to deliver a full load at the maximum frequency allowed, then it could only satisfy $Qf/w(S) < 1$ of the total demand of S. A partition $\{S_1, S_2, \ldots, S_m\}$ is said to be *feasible* if $w(S_k) \leq Qf$ for each $k = 1, 2, \ldots, m$.

Let $L^*(S)$ denote the length of an optimal traveling salesman tour through the set of retailers $S \subseteq N$. Let $T(\kappa)$, with $\kappa \geq 1$, be a κ-*approximation* algorithm for the Traveling Salesman Problem (TSP). We will sometimes simply write T for the heuristic. Let $L^T(S)$ denote the length of a traveling salesman tour through $S \subseteq N$ generated by T. Thus, $L^T(S) \leq \kappa L^*(S)$ for any set $S \subseteq N$.

We will use the following standard notation (see, e.g., [18]). For any sequence of random variables X_n, $n = 1, 2, \ldots$, we write $\lim_{n\to\infty} X_n = X$ (a.s.) to mean $P(\lim_{n\to\infty} X_n = X) = 1$. Replacing lim with $\overline{\lim}$ or $\underline{\lim}$ in all places in this definition provides the other cases.

The analysis of the proposed heuristic is based on the following probabilistic model which is the same as in [3, 9] and similar to [1, 5, 7]. Each retailer is characterized by two random variables: its location and its demand rate which are assumed independent. The retailers are assumed to be located on the Euclidean plane, according to distribution μ which has compact support $\mathcal{A} \subset \mathbb{R}^2$. The depot is placed at the origin and we let $\|x\|$ represent the Euclidean distance between $x \in \mathbb{R}^2$ and the depot. Define $\rho \stackrel{\text{def}}{=} \max\{\|x\| : x \in \mathcal{A}\}$. Let d_i denote retailer i's distance from the depot; i.e., if retailer i is located at $x_i \in \mathbb{R}^2$, then $d_i \stackrel{\text{def}}{=} \|x_i\| \le \rho$. We denote by ψ the distribution of retailer demand rates, and without loss of generality assume the range of feasible rates is $(0, Qf]$.

Since for fixed n the number of partitions of N is finite and the number of different sequences with which the retailers in each subset can be visited is finite, there are only a finite number of FPPs (assuming that each subset is ordered at its optimal reorder interval). Therefore, we let Z_n^{FP} denote the cost of an optimal FPP. A heuristic HFP that constructs an FPP is said to be an *asymptotic ξ-approximation* for Z_n^{FP} (for $\xi \ge 1$) if $\overline{\lim}_{n\to\infty}(1/n)Z_n^{\text{HFP}} \le \xi \cdot \underline{\lim}_{n\to\infty}(1/n)Z_n^{\text{FP}}$ (a.s.).

Next we describe some features of the Bin-Packing Problem (BPP) which we use here. The BPP is defined by a set of item sizes and bins of fixed capacity. In our context the item sizes are retailer demand rates $\{w_i\}$ and the bin capacities are the maximum retailer demand rate that can be served by one vehicle, i.e., Qf. The goal in the BPP is to assign the items to the minimum number of bins without violating the capacity restriction. A set $S \subseteq \{1, 2, \ldots, n\}$ makes up a feasible bin if and only if $w(S) \le Qf$. We let $b^*(S)$ ($b^H(S)$) denote the number of bins used in an optimal solution (a solution that is generated by H) on the set S. Excellent surveys of this problem appear in [11, 12].

An important result concerns the relationship between the number of items and the number of bins required when n is large. Let b_n^* be the number of bins used in the optimal solution to the problem defined by the item sizes $\{w_1, w_2, \ldots, w_n\}$, and assume these item sizes are drawn from the distribution ψ on $(0, Qf]$, with mean $\mathsf{E}[w] = \beta Qf$ and $\beta \in (0, 1]$. Clearly $b_n^* \ge \sum_{i=1}^n w_i/Qf$ which, by the strong law of large numbers, leads to the inequality: $\underline{\lim}_{n\to\infty} b_n^*/n \ge \mathsf{E}[w]/Qf = \beta$, almost surely. Using the techniques of Kingman [21] (see also Rhee and Talagrand [24]) it is possible to show that for each distribution ψ there exists a constant $\gamma \in [\beta, 1]$, called the *bin-packing constant associated with ψ*, such that $\lim_{n\to\infty} b_n^*/n = \gamma$, almost surely. The constant γ is the long-run average number of bins used per item in an optimal packing. Another measure which will be of interest here is the *packing efficiency of ψ* defined as $\alpha \stackrel{\text{def}}{=} \gamma/\beta$. In general $\beta \le \gamma < 2\beta$ and $\alpha \in [1, 2)$. Distributions with $\gamma = \beta$ (or $\alpha = 1$) are said to allow *perfect packing* since asymptotically the amount of wasted space becomes a smaller and smaller fraction of the total bin space used.

The following function plays a crucial role in our analysis. For any $L \ge 0$ and $W > 0$, let

$$z(L, W) \stackrel{\text{def}}{=} \min\left\{\frac{L + c}{t} + \frac{1}{2}htW : \frac{1}{f} \le t \le \frac{Q}{W}\right\}.$$

The function z gives the minimal cost per unit time for serving a set S with fixed cost $L + c$ and total demand rate W. Let t^* denote the value of t achieving the minimum in $z(L, W)$. Note that the vehicle leaves the depot carrying $Wt^* \le Q$ units of product.

Anily and Bramel [3] provide deterministic and probabilistic lower bounds on the cost of any FPP. The rest of this section presents some of those results which are needed for the current analysis. For his sake, first partition the $(L + c)$-axis into three disjoint sets Ω_1, Ω_2, and Ω_3, where

$$\Omega_1 = \left[0, \frac{Qh}{4f}\right), \qquad \Omega_2 = \left[\frac{Qh}{4f}, \frac{Qh}{f}\right), \qquad \text{and} \qquad \Omega_3 = \left[\frac{Qh}{f}, +\infty\right).$$

These sets define three *rings* around the depot; i.e., let $A_j \stackrel{\text{def}}{=} \{x \in \mathcal{A} : 2\|x\| + c \in \Omega_j\}$ for $j \in \{1, 2, 3\}$. For $j \in \{1, 2, 3\}$, let $\bar{d}_j$ denote the retailer's average distance from the depot in the ring A_j. The next theorem is from [3].

THEOREM 1: Under the assumptions of our probabilistic model:

$$\varliminf_{n\to\infty} \frac{1}{n} Z_n^{\text{FP}} \geq \underline{Z} \stackrel{\text{def}}{=} \mu(A_1)\left[\frac{\beta h Q}{2} + \gamma(2\bar{d}_1 + c)f\right] + \mu(A_2)\left[(2\beta - \gamma)\left((2\bar{d}_2 + c)f + \frac{hQ}{2}\right)\right.$$
$$\left. + 2\sqrt{hQf}(\gamma - \beta) \cdot \mathsf{E}[\sqrt{2d + c} \mid 2d + c \in \Omega_2]\right] + \mu(A_3)[(2\bar{d}_3 + c)\beta f + \gamma hQ/2] \qquad \text{(a.s.).} \quad (1)$$

The per-retailer charge defined next will be useful in the determination of the worst-case bound.

LEMMA 2: Define

$$\underline{c}(d) = \begin{cases} hQ\beta/2 + (2d + c)f\gamma, & \text{if } 2d + c \in \Omega_1 \\ (2\beta - \gamma)((2d + c)f + hQ/2) + 2(\gamma - \beta)\sqrt{(2d + c)hQf}, & \text{if } 2d + c \in \Omega_2 \\ (2d + c)f\beta + \gamma hQ/2, & \text{if } 2d + c \in \Omega_3. \end{cases}$$

Then, $\underline{Z} \stackrel{\text{def}}{=} \overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \underline{c}(d_i) \leq \varliminf_{n\to\infty} \frac{1}{n} Z_n^{\text{FP}}$ (a.s.).

The following theorem, proven in [2], characterizes the asymptotic effectiveness of $\underline{Z}$:

THEOREM 3: $\underline{Z}$ is an asymptotic 98.5%-effective lower bound on the average cost per retailer in the best FPP, i.e., $\varliminf_{n\to\infty} \frac{1}{n} Z_n^{\text{FP}} \geq \underline{Z} \geq (0.985) \cdot \overline{\lim}_{n\to\infty} \frac{1}{n} Z_n^{\text{FP}}$ (a.s.).

This implies that as the number of retailers increases, the value of the lower bound approaches at least 98.5% of the average cost per retailer in the best FPP. To prove this result, the authors use the function $\bar{z}(L, W)$, defined below, to bound the cost per bin ($z(L, W)$) from above. This function, which is parameterized by $\theta \in [1, 2)$, will be useful in this paper as well. Define

$$\Omega_1' \stackrel{\text{def}}{=} \left[0, \frac{Qh}{2f\theta}\right), \qquad \Omega_2' \stackrel{\text{def}}{=} \left[\frac{Qh}{2f\theta}, \frac{Qh\theta}{2f}\right], \qquad \text{and} \qquad \Omega_3' \stackrel{\text{def}}{=} \left(\frac{Qh\theta}{2f}, +\infty\right).$$

This partition induces three disjoint rings around the depot which are given by $A_j' = \{x \in \mathcal{A} : 2\|x\| + c \in \Omega_j'\}$ for $j \in \{1, 2, 3\}$. For simplicity in the notation, we will omit the dependence of the sets Ω_j' and A_j' on θ. The following lemma is proven in [3].

LEMMA 4: Fix $\theta \in [1, 2)$. For any $L \geq 0$ and $W \in [0, Qf]$,

$$z(L, W) \le \bar{z}_\theta(L, W) \stackrel{\text{def}}{=} \begin{cases} \dfrac{hW}{(2f)} + (L+c)f, & \text{if } L + c \in \Omega_1', \\ W\sqrt{\dfrac{(L+c)h\theta}{2Qf}} + \sqrt{\dfrac{(L+c)hQf}{2\theta}}, & \text{if } L + c \in \Omega_2', \\ \dfrac{(L+c)W}{Q} + \dfrac{hQ}{2}, & \text{if } L + c \in \Omega_3'. \end{cases}$$

3. A FIXED PARTITIONING SCHEME

In this section, we present a polynomial-time fixed partitioning scheme. In later sections we analyze the performance of the proposed heuristic and prove that it generates a solution that is guaranteed to be asymptotically within 1.5% + ϵ of the cost of the lower bound $\underline{Z}$, for arbitrary $\epsilon > 0$.

The FPP we construct is called *Circular Fixed Partition* (CFP) and its cost on n retailers is denoted Z_n^{CFP}. This partitioning scheme is based on a circular region partitioning scheme similar to those described by Haimovich and Rinnooy Kan [16] (in the context of the capacitated vehicle routing problem), or Anily and Federgruen [5, 7] and Anily [1] for the IRP with split demands. However, in contrast to those papers, here the number of retailers in each of the clusters is not bounded by a constant, but rather is an increasing function of n. (We note that the number of retailers in each cluster is the same, with the possibly exception of one cluster.) Each cluster is then partitioned into subsets of retailers, possibly overlapping, by applying a bin-packing heuristic H. Each subset of retailers created by H is served on a separate route. To emphasize the dependence of CFP on the bin-packing heuristic H we use CFP(H) and $Z_n^{\text{CFP(H)}}$ for CFP and Z_n^{CFP}, respectively.

We now describe the proposed partitioning scheme. In what follows, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

Circular Fixed Partitioning Scheme (CFP(H))

- *Step 1:* Partition the circle with radius ρ by means of radial cuts into $\lfloor n^{1/4} \rfloor$ consecutive *sectors* each containing $\lfloor n^{5/8} \rfloor \lfloor n^{1/8} \rfloor$ retailers, and possibly one additional sector containing $n' \stackrel{\text{def}}{=} n - \lfloor n^{1/4} \rfloor \lfloor n^{5/8} \rfloor \lfloor n^{1/8} \rfloor$ retailers. Let K denote the number of sectors generated, i.e., $K = \lfloor n^{1/4} \rfloor$ or $K = \lfloor n^{1/4} \rfloor + 1$. If $K = \lfloor n^{1/4} \rfloor + 1$, then let sector K be the one that contains n' retailers. Let S_k denote the set of retailers in sector k, for $k = 1, 2, \ldots, K$.
- *Step 2:* For each $k = 1, 2, \ldots, K$, partition the sector k, starting from its outside boundary, by circular cuts such that each of the *clusters* obtained contains exactly $p =^{\text{def}} \lfloor n^{1/8} \rfloor$ retailers, except for the inner cluster in sector K that may contain less than p retailers. (Note that, in each of the sectors $k = 1, 2, \ldots, K - 1$, we use exactly $q = \lfloor n^{5/8} \rfloor - 1$ circular cuts. In sector K we use $q' = \lceil |S_K|/p \rceil - 1$ circular cuts.)
- *Step 3:* Let $C_{j,\ell}$ denote the retailers in the ℓth cluster generated in A_j', for $j \in \{1, 2, 3\}$, where a cluster is considered in A_j' if its closest point is in A_j'. For each cluster $C_{j,\ell}$ choose an arbitrary point $v_{j,\ell}$ on the circular cut enclosing the cluster from below. Note $v_{j,\ell} \in A_j'$ for all ℓ. In each cluster, apply the bin-packing heuristic H to pack the demand rates of the retailers in the cluster into bins of capacity Qf. Let $\mathcal{B}_{j,\ell}$ be the resulting set of bins in cluster $C_{j,\ell}$. Note $b^H(C_{j,\ell}) = |\mathcal{B}_{j,\ell}|$. Each bin is served on a separate route. To determine the exact sequence in which the retailers

in a bin $B \in \mathcal{B}_{j,\ell}$ are visited on their route, we apply the heuristic $T(\kappa)$ to the set of retailers in B and the point $v_{j,\ell}$. Connect the tour to the depot by adding two copies of the arc connecting the depot to $v_{j,\ell}$. The bin B is served every t^* units of time, where t^* is the minimizer in $z(2\|v_{j,\ell}\| + L^T(B \cup \{v_{j,\ell}\}), w(B))$.

Note that Steps 1–2 of CFP produce a region partitioning such that all generated clusters are contained within a single sector and their boundary consists of two circular cuts and two radial cuts.

We now analyze the computational complexity of CFP(H) when applied to a set of n retailers. Step 1 of the heuristic requires ordering the retailers according to their angular coordinate. Step 2 requires ordering the retailers within each sector according to their radial distance. Therefore, the complexity of Steps 1–2 is $O(n \log n)$. The complexity of Step 3 depends on the complexity of the bin-packing heuristic H and the traveling salesman heuristic T. Suppose n items can be packed using H in time $h(n)$, and n points can be routed using T in time $t(n)$. Then, Step 3's complexity is $O(n^{7/8} \cdot \max\{h(n^{1/8}), t(n^{1/8})\})$. Therefore, the complexity of the whole algorithm is $O(\max\{n \log n; n^{7/8} \cdot \max\{h(n^{1/8}), t(n^{1/8})\}\})$. In Section 5.1 we present a bin-packing heuristic whose complexity is linear in the number of items. If we also use the simple 2-approximation minimum spanning tree algorithm for the TSP (see [22]), then $t(n) = O(n^2)$ and the CFP has overall complexity $O(n^{9/8})$. If instead we use the 1.5-approximation Christofides' heuristic (see [10]), which has complexity $t(n) = O(n^3)$, then the CFP has overall complexity $O(n^{1.25})$.

4. A PERFORMANCE ANALYSIS OF CFP

In this section we analyze the quality of the policy produced by CFP. For that sake we first need a few known results on the classical vehicle routing problem and RPs. In general, let X denote a set of m points in the Euclidean plane within a circle of radius ρ from the origin. An RP partitions the circle into disjoint subregions $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_R$ such that each retailer in X belongs to exactly one of the regions. As defined above, let $L^*(X)$ be the length of the optimal traveling salesman tour through the set X. Let also $\|L^*(X) \cap \tilde{X}_i\|$ be the total length of those parts of the optimal traveling salesman tour through X that are within the subregion $\tilde{X}_i$. That is, $\sum_{i=1}^{R} \|L^*(X) \cap \tilde{X}_i\| = L^*(X)$. The first lemma is immediate from Theorem 3 in Karp [20].

LEMMA 5: Let $\mathcal{X} = \{X_1, \ldots, X_R\}$ be a partition of X generated by a RP. Let $\tilde{X}_i$, $i = 1, 2, \ldots, R$, denote the subregions and Π_i^* its perimeter. Then, $L^*(X_i) \le \|L^*(X) \cap \tilde{X}_i\| + 1.5\Pi_i^*$, and therefore $\sum_{i=1}^{R} L^*(X_i) \le L^*(X) + 1.5\Pi^*$, where $\Pi^* \stackrel{\text{def}}{=} \sum_{i=1}^{R} \Pi_i^*$.

The length of the optimal traveling salesman tour ($L^*(X)$) is $O(\rho\sqrt{m})$ as follows, e.g., from the following lemma, due to [16].

LEMMA 6: If X is contained in a connected planar region with area A and finite perimeter $\hat{\Pi}$, then $L^*(X) \le \sqrt{2mA} + 1.5\hat{\Pi}$.

In subsequent analysis we use both lemmas to bound the cost of the solution generated by CFP. The CFP generates a set of bins $\cup_{j,\ell} \mathcal{B}_{j,\ell}$ that is a partition of N and also each bin B in $\mathcal{B}_{j,\ell}$ consists of retailers that are contained in the cluster $C_{j,\ell}$. The route through the retailers in $B \subseteq C_{j,\ell}$ starts at the depot, passes through $v_{j,\ell}$, visits all retailers in B, and goes back to the depot through $v_{j,\ell}$. We say that the retailers in bin B and the point $v_{j,\ell}$ induce a *district*. We define the set N' to consist of the set of retailers N as well as $b^H(C_{j,\ell})$ copies of the point $v_{j,\ell}$ for any cluster $C_{j,\ell}$. Clearly, $|N'| \leq 2n$. We note that the set of districts is a partition of N'. Let $\tilde{\Pi}$ be the total perimeter of the clusters generated by Steps 1–2 of CFP. The proof of the following lemma is given in the Appendix.

LEMMA 7:

(a) The number of retailers in sector k, for $k \leq K - 1$, at the end of Step 1 of CFP is $\lfloor n^{5/8} \rfloor \lfloor n^{1/8} \rfloor$. The number of retailers in sector K is bounded from above by $n^{7/8} + n^{3/4} - n^{5/8} + n^{3/8} - n^{1/4} - n^{1/8} + 1$.
(b) For any $n \geq 2$, the total perimeter $\tilde{\Pi}$ of the generated clusters at the end of Step 2 of the CFP is at most $\rho[4\pi(n^{3/4} + 2n^{5/8} + n^{3/8} + \sqrt{n} + n^{1/8}) + (8\pi + 2)n^{1/4} + 50\pi + 2] = O(n^{3/4})$.

For $j \in \{1, 2, 3\}$, let $\mathcal{C}'_j$ be the index set of the clusters created by CFP whose closest point is in A'_j; i.e., if $\ell \in \mathcal{C}'_j$, then $v_{j,\ell} \in A'_j$. Also let $B_{j,\ell,i}$ denote the ith bin of $\mathcal{B}_{j,\ell}$. The total cost of the generated solution is given by

$$Z_n^{\text{CFP(H)}} = \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} z(2\|v_{j,\ell}\| + L^T(B_{j,\ell,i} \cup \{v_{j,\ell}\}), w(B_{j,\ell,i})).$$

In the sequel we develop an upper bound on $Z_n^{\text{CFP(H)}}$.

It is easy to show (see, e.g., [3]) that, for any $L \geq 0$, $r > 0$, and $W > 0$, $z(L, W) < z(L + r, W) \leq z(L, W) + fr$. Thus for any j, ℓ and bin $B_{j,\ell,i} \subseteq C_{j,\ell}$,

$$z(2\|v_{j,\ell}\| + L^T(B_{j,\ell,i} \cup \{v_{j,\ell}\}), w(B_{j,\ell,i})) \leq z(2\|v_{j,\ell}\|, w(B_{j,\ell,i})) + fL^T(B_{j,\ell,i} \cup \{v_{j,\ell}\}).$$

Hence,

$$Z_n^{\text{CFP(H)}} \leq \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} z(2\|v_{j,\ell}\|, w(B_{j,\ell,i})) + f \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} L^T(B_{j,\ell,i} \cup \{v_{j,\ell}\}). \tag{2}$$

As for other vehicle routing problems, the first term of the upper bound [in (2)] is dominant as n increases. Let $\mathcal{L}_n$ be the second term of (2), i.e., $\mathcal{L}_n \stackrel{\text{def}}{=} f \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} L^T(B_{j,\ell,i} \cup \{v_{j,\ell}\})$. The following lemma characterizes the growth rate of $\mathcal{L}_n$.

LEMMA 8: $\mathcal{L}_n = O(n^{3/4})$, i.e., $\overline{\lim}_{n\to\infty}(\mathcal{L}_n/n^{3/4}) < +\infty$ (a.s.).

PROOF: Since $T = T(\kappa)$ is a κ-approximation algorithm for the TSP, we obtain

$$\mathcal{L}_n = f \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} L^{T(\kappa)}(B_{j,\ell,i} \cup \{v_{j,\ell}\}) \leq \kappa f \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} L^*(B_{j,\ell,i} \cup \{v_{j,\ell}\}).$$

Note that $\cup_i(\mathcal{B}_{j,\ell,i} \cup \{v^i_{j,\ell}\})$, where $v^i_{j,\ell} \stackrel{\text{def}}{=} v_{j,\ell}$ for any $\mathcal{B}_{j,\ell,i} \subseteq \mathcal{C}_{j,\ell}$ is a region partition of N'. In order to use Lemma 5, we need to bound $\mathcal{L}_n$ in terms of the optimal traveling salesman tours in the clusters. For that sake, we note that the sum of the tours via the districts in a single cluster is bounded from above by the optimal tour over the clusters times the number of districts which is at most $n^{1/8}$. That is, for any cluster $C_{j,\ell}$, the following inequality holds:

$$\sum_{i \in \mathcal{B}_{j,\ell}} L^*(B_{j,\ell,i} \cup \{v_{j,\ell}\}) \le n^{1/8} L^*(\cup_i(B_{j,\ell,i} \cup \{v_{j,\ell}\})).$$

Therefore, using Lemma 5, we obtain

$$\mathcal{L}_n \le \kappa f \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} n^{1/8} L^*(\cup_{i \in \mathcal{B}_{j,\ell}} B_{j,\ell,i} \cup \{v_{j,\ell}\}) \le \kappa f n^{1/8} L^*(N') + 1.5\tilde{\Pi},$$

where $\tilde{\Pi}$ is the total perimeter of the clusters generated by Steps 1–2 of CFP. Combining this bound with Lemmas 6 and 7, we obtain

$$\mathcal{L}_n \le \kappa f n^{1/8}(\sqrt{2(2n)\pi\rho^2} + 2\pi\rho) + 1.5\kappa f n^{1/8}\tilde{\Pi} \le 2\kappa f \rho \sqrt{\pi}\, n^{5/8} + 2\pi\rho\kappa f n^{1/8} + 1.5\kappa f n^{1/8}\tilde{\Pi}.$$

Since $\tilde{\Pi}$ is $O(n^{3/4})$ from Lemma 7 part (b), we have $\mathcal{L}_n = O(n^{7/8})$. □

We now apply Lemma 4 to (2):

$$Z_n^{\text{CFP(H)}} \le \sum_{j=1}^{3} \sum_{\ell \in \mathcal{C}'_j} \sum_{i \in \mathcal{B}_{j,\ell}} \bar{z}_\theta(2\|v_{j,\ell}\|, w(B_{j,\ell,i})) + \mathcal{L}_n. \tag{3}$$

The analysis of the first term of this upper bound is more involved. Clearly, the bin-packing heuristic H plays a central role in CFP(H). In order to quantify the effectiveness of H, we need the following terms: let b_n^H denote the number of bins used by H to pack a set of n demand rates $\{w_1, \ldots, w_n\}$ drawn independently from ψ into bins of capacity Qf. Note that for a given n, b_n^H is a random variable that is bounded from above by n. We define $\hat{\gamma}^H \stackrel{\text{def}}{=} \overline{\lim}_{n\to\infty} b_n^H/n$, almost surely. Note $\hat{\gamma}^H$ could be a random variable. Define $\gamma^H \stackrel{\text{def}}{=} \sup \hat{\gamma}^H$, where $\sup X =^{\text{def}} \sup\{x : \text{P}(X = x) > 0\}$ for any random variable X. Then γ^H is defined as the *bin-packing constant of* H (associated with ψ). For any bin-packing heuristic H we have: $\beta \le \gamma \le \gamma^H \le 1$. Define $\alpha^H \stackrel{\text{def}}{=} \gamma^H/\beta$. Note that any *reasonable* bin-packing heuristic will have $\alpha^H < 2$. The bin-packing heuristic H is said to be *asymptotically optimal for* ψ if $\gamma^H = \gamma$ or, equivalently, if $\alpha^H = \alpha$.

In CFP(H) the bin-packing heuristic H is applied on the demand rates of the retailers within each cluster. The number of clusters grows with n, but at a slower rate [at a rate $o(n)$], and thus the number of retailers within each cluster grows at a rate slower than n. It is important to understand whether the packing of the n retailers that results from this "partitioning" method is as asymptotically efficient as when H is applied *directly* to the n retailers. This property is required for our results on the CFP(H) and essentially amounts to a requirement that the convergence rate of H be "fast enough" relative to the *partitioning rate* $a(n)$ (which in this case is $n^{7/8}$). Therefore, any bin-packing heuristic that is used as a subroutine in CFP must be, what we call, *robust-under-partitioning,* or more specifically, robust-under-*sublinear*-partitioning, which we next state formally:

PROPERTY 9: Suppose n items are partitioned, independently of their sizes, into $a(n) = o(n)$ clusters denoted $C_1, C_2, \ldots, C_{a(n)}$. In each cluster the bin-packing heuristic H is applied. The heuristic H is said to be *robust-under-partitioning* if for any sequence of finite nonnegative and uniformly bounded constants[1] $\{\vartheta_\ell\}$, $\ell \geq 1$, we have

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell b^H(C_\ell) \leq \gamma^H \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell |C_\ell| \qquad \text{(a.s.).} \tag{4}$$

In particular, if $\vartheta_\ell = 1$, for all $\ell \geq 1$, robustness-under-partitioning says that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\ell=1}^{a(n)} b^H(C_\ell) \leq \gamma^H \qquad \text{(a.s.).}$$

This implies that the packing (of the n retailers) that results is asymptotically as efficient as if H were applied directly to the n retailers. Or put another way, the number of extra bins required, because of the partitioning of the items, is negligible (asymptotically).

From here on, we assume that the bin-packing heuristic H (used in Step 3 of CFP) is robust-under-partitioning. In the next section we will describe one such heuristic.

We now bound from above the asymptotic cost of the policy constructed by CFP(H). Let $\bar{d}'_j$ denote the expected distance to the depot in ring A'_j, for $j \in \{1, 2, 3\}$. The proof of the following lemma is given in the Appendix.

LEMMA 10:

$$\overline{\lim_{n\to\infty}} \frac{1}{n} Z_n^{\text{CFP(H)}} \leq \bar{Z}^H \stackrel{\text{def}}{=} \mu(A'_1)\left[(2\bar{d}'_1 + c)\gamma^H f + \frac{\beta h Q}{2}\right] + \mu(A'_2)\sqrt{2hQf\beta\gamma^H} \cdot \mathsf{E}[\sqrt{2d + c}|2d + c \in \Omega'_2] + \mu(A'_3)\left[(2\bar{d}'_3 + c)\beta f + \frac{\gamma^H h Q}{2}\right] \qquad \text{(a.s.).} \tag{5}$$

The following per-retailer charge will be useful in proving the asymptotic worst-case bound.

THEOREM 11: Define:

$$\bar{c}^H(d) = \begin{cases} \dfrac{hQ\beta}{2} + (2d + c)f\gamma^H, & \text{if } 2d + c \in \Omega'_1, \\ \sqrt{2hQf\gamma^H\beta(2d + c)}, & \text{if } 2d + c \in \Omega'_2, \\ (2d + c)\beta f + \dfrac{\gamma^H hQ}{2}, & \text{if } 2d + c \in \Omega'_3. \end{cases}$$

[1] The constants $\{\vartheta_\ell\}$, $\ell \geq 1$, are also assumed to be independent of the sizes of the items in the clusters.

Then

$$\underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{i\in N} \bar{c}^H(d_i) = \bar{Z}^H \geq \overline{\lim}_{n\to\infty} \frac{1}{n} Z_n^{\mathrm{CFP(H)}}.$$

PROOF: The equality follows since $\underline{\lim}_{n\to\infty}(1/n)\ \sum_{i=1}^n\ \bar{c}^H(d_i)$ is, a.s., equal to $\bar{Z}^H$ [the right-hand side of (5)] by the strong law of large numbers. The inequality is from Lemma 10. □

In order to prove the asymptotic worst-case bound, we compare the per-retailer cost allocations from the upper bound on CFP (Theorem 11) to the one from the lower bound (Lemma 2). To make this comparison, we need the following lemma:

LEMMA 12: Define

$$\xi^H \stackrel{\mathrm{def}}{=} \max\left\{\frac{\bar{c}^H(\|x\|)}{\underline{c}(\|x\|)} : x \in \mathcal{A}\right\};$$

then

$$\overline{\lim}_{n\to\infty} \frac{1}{n} Z_n^{\mathrm{FP}} \leq \overline{\lim}_{n\to\infty} \frac{1}{n} Z_n^{\mathrm{CFP(H)}} \leq \bar{Z}^H \leq \xi^H \cdot \underline{\lim}_{n\to\infty} \frac{1}{n} Z_n^{\mathrm{FP}};$$

i.e., $\frac{1}{n} Z_n^{\mathrm{CFP(H)}}$ and $\bar{Z}^H$ are asymptotic ξ^H-approximations on the average cost per retailer in the best FPP.

PROOF: First note that $\bar{c}^H(\|x\|) \leq \xi^H \cdot \underline{c}(\|x\|)$ for any $x \in \mathcal{A}$. The result then follows by using the definitions of the $\bar{c}^H$ and $\underline{c}$. □

Define $r^H =^{\mathrm{def}} \gamma^H/\gamma$, i.e., r^H is the *asymptotic worst-case ratio* of the bin-packing heuristic H. For example, if H is asymptotically optimal, then $r^H = 1$. Note $r^H \in [1, 2)$. The following theorem is proven in the Appendix.

THEOREM 13:

(a) Suppose we apply CFP(H) with a bin-packing heuristic H that is robust-under-partitioning and whose asymptotic worst-case ratio is r^H. Then, CFP(H) is an asymptotic $1.015\sqrt{r^H}$-approximation algorithm for the best FPP.
(b) If H is asymptotically optimal, then CFP(H) is an asymptotic 1.015-approximation algorithm for the best FPP.

REMARK: We note here the impact of using a bin-packing heuristic H which is not robust (as defined in Property 9). Say a heuristic is *τ-robust-under-partitioning* if (4) holds only when the right-hand side is multiplied by $\tau \geq 1$. Then the details of the analysis change, but the general idea stays the same. One can show that the CFP(H) is then an asymptotic $1.015\sqrt{\tau r^H}$-approximation for the best FPP. (In particular, choose $\theta = \tau\alpha^H$.)

In the next section we show that, for arbitrary $\epsilon > 0$, the Set-Covering based bin-packing heuristic is an asymptotic $(1 + \epsilon)$-approximation for the BPP and is $(1 + \epsilon)$-robust-under-partitioning. According to Theorem 13, using this heuristic in CFP results in an FPP whose average cost is asymptotically within at most $1.5\% + \epsilon'$ of the best FPP, for some $\epsilon' > 0$ (where $\epsilon' \to 0$ as $\epsilon \to 0$). It is not clear *a priori* whether the 1.5% gap is due to weakness of the lower bound or of CFP. As shown in [3] the lower bound (1) is not always asymptotically tight with the optimal solution. Indeed, Anily and Bramel [3] present an example where for $\alpha = \sqrt{2}$ the lower bound is indeed 1.5% below the best FPP's cost. This holds for any number of retailers. In our worst-case analysis of CFP (the proof of Theorem 13), we see that if $\gamma^H = \gamma$ then the worst situation occurs again when $\alpha = \sqrt{2}$. So the performance of CFP might actually be better than what we have proved here. It remains an open question whether the CFP using an asymptotic optimal bin-packing heuristic as a subroutine produces an FPP that is asymptotic optimal within the FPP class. However, our results here come very close (within 1.5%) to answering this question in the affirmative.

5. A ROBUST-UNDER-PARTITIONING BIN-PACKING HEURISTIC

We now describe the bin-packing heuristic initially developed by Karmarkar and Karp [19] and analyzed by Federgruen and van Ryzin [14]. In subsection 5.1 we describe the heuristic and some of its properties as proven in [14]. The main property is that, for arbitrary $\epsilon > 0$, it is an asymptotic $(1 + \epsilon)$-approximation for the bin-packing problem under general conditions on the distribution of item sizes (ψ). In Subsection 4.2 we prove that the heuristic is $(1 + \epsilon)$-robust-under-partitioning. Therefore, using this bin-packing heuristic as a subroutine in CFP results in an FPP whose cost asymptotically is at most $1.5\% + \epsilon'$ more than the cost of an optimal FPP.

In the description of this scheme bins are of capacity Qf and item sizes (demand rates) are in the range $(0, Qf]$ and are distributed according to ψ.

5.1. The Set-Covering Based Bin-Packing Heuristic

This heuristic is based on solving a linear programming relaxation of the set-covering formulation of the BPP where the item sizes are discretized: Pick an integer $I > 0$ and divide the range $(0, Qf]$ into I subintervals of size $\delta = Qf/I$ each (δ is the *discretization level*). We denote the heuristic by $H(\delta)$. Let $u_i =^{\text{def}} i \cdot \delta$ for $i = 0, 1, \ldots, I$. Round up the size of each item to the next discrete point in $\{u_1, u_2, \ldots, u_I\}$. Then formulate the BPP as a set-covering problem with I rows (one for each potential item size) and a column for each feasible *bin configuration.* A bin configuration is a specification of the number of items of each type (each discrete size) that makes up a feasible bin. The heuristic solution to the BPP is obtained by solving this set-covering problem *as a linear program* and rounding up each nonzero element of the solution vector to the nearest integer.

Federgruen and van Ryzin [14] prove that $\gamma^{H(\delta)}$, the bin-packing constant associated with $H(\delta)$, converges to γ, the bin-packing constant, as $\delta \to 0$. That is, as the discretization becomes finer, the algorithm gets closer to being asymptotically optimal. The drawback of this heuristic is its complexity which is a function of Qf/δ, the number of intervals into which $[0, Qf]$ is partitioned.

We define now a few terms that are needed for the description of the algorithm and for the analysis thereafter. Let $\mathcal{I}_i$ denote the interval $(u_{i-1}, u_i]$, for $i = 1, 2, \ldots, I$. In the heuristic, each item size in $\mathcal{I}_i$ is rounded up to u_i. For each $i = 1, 2, \ldots, I$, let $\bar{\psi}_i(\delta) = \mathsf{P}(w \in \mathcal{I}_i)$, i.e., $\bar{\psi}_i(\delta)$ designates the probability that an item is rounded up to u_i, and define $\bar{\psi}(\delta) = (\bar{\psi}_1(\delta),$

$\bar{\psi}_2(\delta), \ldots, \bar{\psi}_I(\delta))$. We also define $\underline{\psi}_i(\delta) = \mathsf{P}(w \in \mathcal{I}_{i+1})$ for each $i = 0, 1, 2, \ldots, I-1$, and let $\underline{\psi}_I(\delta) = 0$ and $\underline{\psi}(\delta) = (\underline{\psi}_1(\delta), \underline{\psi}_2(\delta), \ldots, \underline{\psi}_I(\delta))$. For any nonempty set S of items drawn from ψ, and for each $i = 1, 2, \ldots, I$, define $\bar{n}_i(S) \stackrel{\text{def}}{=} |\{k : k \in S \text{ and } w_k \in \mathcal{I}_i\}|$, i.e., the number of items in S that get rounded up to u_i. Also define $\bar{n}(S) = (\bar{n}_1(S), \bar{n}_2(S), \ldots, \bar{n}_I(S))$. Note $\bar{n}(S)$ is a vector of random variables. In what follows, for any vector $x = (x_1, x_2, \ldots, x_I)$, let $|x| \stackrel{\text{def}}{=} \sum_{i=1}^{I} |x_i|$, i.e., the L_1-norm.

Federgruen and van Ryzin [14] first show the following continuity result:

LEMMA 14: If ψ is bounded and continuous almost everywhere, then for every $\epsilon > 0$ there exists a $\Delta > 0$ such that for all $\delta \in (0, \Delta]$: $|\bar{\psi}(\delta) - \underline{\psi}(\delta)| < \epsilon$.

From here on we assume that ψ is bounded and continuous almost everywhere on $(0, Qf]$.

To construct the set-covering formulation, we define the matrix A consisting of I rows, one row for each of the possible discrete item sizes, and J columns, one for each feasible bin configuration. The entry a_{ij} (for $i = 1, 2, \ldots, I$ and $j = 1, \ldots, J$) denotes the number of items of size u_i in bin configuration j. Note that since we have excluded null items (items of size 0) the number of feasible bin configuration is finite (it depends on $\delta > 0$). Let $b^{IP}(\bar{n}(S))$ denote the value of the following integer programming (IP) set-covering formulation of the bin-packing problem:

$$b^{IP}(\bar{n}(S)) = \min\{e^{\mathsf{T}}y : Ay \geq \bar{n}(S), y \geq 0 \text{ and integer}\},$$

where e represents a vector of 1's of appropriate dimension. In this formulation y_j represents the number of bins of type j that are used by the optimal solution; thus $b^{IP}(\bar{n}(S))$ represents the number of bins needed to pack the discretized set of items represented by $\bar{n}(S)$. Clearly, $b^*(S) \leq b^{IP}(\bar{n}(S))$.

In the heuristic, the following linear programming relaxation (LP) of this integer program is solved: Let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_I)$ be a nonnegative vector, and let $b^{LP}(\zeta) = \min\{e^{\mathsf{T}}y : Ay \geq \zeta, y \geq 0\}$. Federgruen and van Ryzin prove that b^{LP} has the following properties:

LEMMA 15:

1. $b^{LP}(v\zeta) = vb^{LP}(\zeta)$ for all $\zeta \geq 0$ and $v \geq 0$.
2. $b^{LP}(\zeta) - |\zeta_0| \leq b^{LP}(\zeta + \zeta_0) \leq b^{LP}(\zeta) + |\zeta_0|$ for all $\zeta \geq 0$ and $\zeta_0 \geq 0$.

After solving the linear programming relaxation $b^{LP}(\bar{n}(S))$, each nonzero element of the solution vector (a bin configuration) is rounded up to the next integer value (the number of times the bin configuration is used in the solution). This heuristic solution value, denoted by $b^{H(\delta)}(S)$, is compared to a lower bound in [14]. For any $\delta > 0$ define: $\gamma_\delta \stackrel{\text{def}}{=} b^{LP}(\underline{\psi}(\delta))$, which is a lower bound on the asymptotic cost per item when items are rounded *down* to the nearest discrete item size. Federgruen and van Ryzin prove the following theorem (see Theorem 4 in [14]):

THEOREM 16:

1. For any $\delta > 0$, $\gamma_\delta \leq \gamma$.
2. $\lim_{\delta \to 0} \gamma_\delta = \gamma$.

The next lemma provides an upper bound on the number of bins used by $H(\delta)$ when applied to a set $S \subseteq N$. (A similar property was also needed in [14].)

LEMMA 17: Given $\delta > 0$, $\forall S \subseteq N$,

$$b^{H(\delta)}(S) \leq |S|(\gamma_\delta + |\bar{\psi}(\delta) - \underline{\psi}(\delta)|) + |\bar{n}(S) - \mathsf{E}[\bar{n}(S)]| + \left\lceil \frac{Qf}{\delta} \right\rceil. \quad (6)$$

PROOF: Consider $H(\delta)$ applied to $S \subseteq N$. The optimal solution $\bar{y}$ to the linear programming relaxation contains at most I nonzero entries, since the number of positive variables in any basic feasible solution to LP cannot exceed the number of rows of the matrix A. Thus, by rounding each nonzero component $\bar{y}_j$ up to the nearest integer, the integer (heuristic) solution generated will cost at most 1 unit more for each of the I positive basic variables. Therefore, we have

$$b^{H(\delta)}(S) \leq b^{LP}(\bar{n}(S)) + I = |S| \cdot b^{LP}(\bar{\nu}(S)) + I,$$

where we have defined $\bar{\nu}_i(S) = \bar{n}_i(S)/|S|$ for $i = 1, 2, \ldots, I$ and $\bar{\nu}(S) = (\bar{\nu}_1(S), \bar{\nu}_2(S) \ldots, \bar{\nu}_I(S))$ and used Lemma 15, part 1. Now, using Lemma 15, part 2, we get

$$b^{LP}(\bar{\nu}(S)) \leq b^{LP}(\bar{\psi}(\delta)) + |\bar{\nu}(S) - \bar{\psi}(\delta)| \quad (7)$$

and

$$b^{LP}(\bar{\psi}(\delta)) \leq b^{LP}(\underline{\psi}(\delta)) + |\bar{\psi}(\delta) - \underline{\psi}(\delta)| = \gamma_\delta + |\bar{\psi}(\delta) - \underline{\psi}(\delta)|. \quad (8)$$

Combining (7) and (8), and since $\bar{\nu}(S) \cdot |S| = \bar{n}(S)$ and $|S| \cdot \bar{\psi}(\delta) = \mathsf{E}[\bar{n}(S)]$, we get

$$b^{H(\delta)}(S) \leq |S|(\gamma_\delta + |\bar{\psi}(\delta) - \underline{\psi}(\delta)|) + |\bar{n}(S) - \mathsf{E}[\bar{n}(S)]| + I.$$

Since $I = \lceil Qf/\delta \rceil$, we get the desired result. □

The asymptotic effectiveness of $H(\delta)$ can now be demonstrated.

LEMMA 18: Given $\epsilon > 0$, there exists a $\Delta > 0$ such that, for all $\delta \in (0, \Delta]$,

$$\gamma \leq \gamma^{H(\delta)} \leq \gamma_\delta + \epsilon \leq \gamma + \epsilon \qquad \text{(a.s.)},$$

where $\gamma^{H(\delta)} \stackrel{\text{def}}{=} \overline{\lim}_{n\to\infty} b_n^{H(\delta)}/n$, almost surely.

PROOF: The first inequality is obvious. The third inequality is a direct result of Lemma 16, part 1. For the middle inequality, recall $\delta > 0$ is fixed. Now using (6), as $|S| \to \infty$, the quantity $(1/|S|)|\bar{n}(S) - \mathsf{E}[\bar{n}(S)]|$ goes to 0 by the strong law of large numbers. As $|S| \to \infty$ the last term of (6) becomes a negligible fraction of $|S|$. Using Lemma 14, we get the desired result. □

Therefore, given $\epsilon > 0$, there exists a $\delta > 0$ such that $H(\delta)$ is an asymptotic $(1 + \epsilon)$-approximation for the BPP. Federgruen and van Ryzin show in [14] that the complexity of $H(\delta)$ is linear in the number of items. Moreover, under the uniform model of computation, it takes constant time since the dimension of the matrix A is independent of the number of items. We refer the reader to [14] for more details on the effectiveness of this particular solution approach to the bin-packing problem. Finally, since we apply the $H(\delta)$ heuristic on sets of $n^{1/8}$ items (retailers) and perform this $n^{7/8}$ times, the CFP heuristic is also linear in the number of retailers.

5.2. Proof of Robustness

Next we prove that given $\epsilon > 0$, there exists a $\delta > 0$ such that $H(\delta)$ is $(1 + \epsilon)$-robust-under-partitioning. Assume the n items are partitioned independently of their sizes into $a(n)$ clusters denoted $\{C_\ell\}_{\ell=1}^{a(n)}$. For $\ell = 1, 2, \ldots, a(n)$, let $n_\ell = |C_\ell|$. We assume that $a(n)$ is an increasing and unbounded function of n and that $a(n) = o(n)$. Also assume we are given a sequence of uniformly bounded non-negative real numbers $\vartheta_\ell \le \bar{\vartheta} < +\infty$, for $\ell \ge 1$.

THEOREM 19: Given $\epsilon > 0$, there exists a $\Delta > 0$ such that, for all $\delta \in (0, \Delta]$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell b^{H(\delta)}(C_\ell) \le (\gamma_\delta + \epsilon) \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell |C_\ell| \qquad \text{(a.s.)}.$$

PROOF: Fix $\epsilon > 0$, and let Δ be as specified by Lemma 14. Then by Lemma 17, for $\delta \le \Delta$, $b^{H(\delta)}(S) \le |S|(\gamma_\delta + \epsilon) + |\bar{n}(S) - \mathsf{E}[\bar{n}(S)]| + \lceil Qf/\delta \rceil$, $\forall S \subseteq N$. Then

$$\frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell b^{H(\delta)}(C_\ell) \le (\gamma_\delta + \epsilon) \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell |C_\ell| + \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell |\bar{n}(C_\ell) - \mathsf{E}[\bar{n}(C_\ell)]| + \frac{\left\lceil \frac{Qf}{\delta} \right\rceil}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell. \tag{9}$$

The last term in (9) disappears since $\sum_{\ell=1}^{a(n)} \vartheta_\ell \le a(n)\bar{\vartheta} = o(n)$ and $\delta > 0$ is fixed. The middle term on the right-hand side of (9) represents the weighted sum of differences of multinomial random variables with their expected values. To analyze this term, we invoke Lemma 21 (proven in the Appendix) which says that the weighted sum of these deviations is a negligible fraction of n. Specifically, Lemma 21 states that, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \qquad \bar{n}(C_\ell) - \mathsf{E}[\bar{n}(C_\ell)]| = 0.$$

Thus this term disappears in the limit and we obtain the desired result. □

Now since $\gamma_\delta \le \gamma$, we see that given $\epsilon > 0$, the heuristic $H(\delta)$ is $(1 + \epsilon/\gamma^H)$-robust-under-partitioning. Thus it is $(1 + \epsilon')$-robust-under-partitioning where $\epsilon' \to 0$ whenever $\epsilon \to 0$.

6. CONCLUSION

In many logistical problems, practitioners are aware of the added cost involved in restricting themselves to simpler policies that are easy to implement. Usually this additional cost is hard to assess relative to the benefit obtained by using simple rules so it is neglected in the overall profit evaluation. This paper as well as [3, 5, 9] enhance the understanding of this issue in some NP-hard IRPs, where the design of an algorithm that solves them to optimality is probably impossible. Moreover, no clue is yet known about the structure of optimal policies for these problems. It is conceivable that optimal policies might be very complex and therefore expensive and unattractive from an implementation point of view. As a result, in the last decade, researchers have concentrated on policy classes that have an appropriate structure to be effective, as for example, the partitioning policies.

Three lower bounds are proposed in the literature for IRPs. The lowest is provided by Chan, Federgruen, and Simchi-Levi [9] that developed a lower bound on the optimal cost under any strategy. Their lower bound is shown to be asymptotically tight with the optimal cost of a partitioning policy for the unsplit demand case for problems that allow perfect packing. Anily and Federgruen [4] restrict themselves to the class of partitioning policies: They provide an asymptotically tight lower bound on the optimal cost of a policy in this class. Anily and Bramel [3] further restrict themselves to FPPs. Thus, they provide the largest lower bound among the three, i.e., a lower bound on the optimal cost over all FPPs. The three lower bounds coincide when perfect packing prevails and when the number of retailers is large enough. In particular, if the lower bounds of Chan, Federgruen, and Simchi-Levi [9] and of Anily and Federgruen [5], that do not depend on any packing features, are close then one can conclude that the restriction to partitioning policies for large scale systems is worthy, in view of their simplicity with respect to logistical issues. On the other hand, as this paper together with [3, 9] demonstrate, the restriction to FPPs may entail a significant loss of profits for problems that are far of allowing perfect packing. In such problems it is worth investigating the potential decrease in cost if the restriction to an FPP is replaced by the restriction to a partitioning policy (for the split demand case).

In Section 3 we construct a heuristic solution in the FPP class, named CFP(H). The heuristic first applies a circular regional partitioning and then a bin-packing heuristic to determine the assignment of the retailers to trucks. As explained at the end of Section 3, if in the heuristic the simple 2-approximation minimum spanning tree algorithm for the TSP and the bin-packing heuristic described in Section 5.1 are used, then the overall complexity of CFP(H) is $O(n^{1.25})$. While this theoretical complexity bound is impressive, further research should investigate both the actual running time and gap between the heuristic solution and the various lower bounds as well as the sensitivity of these measures with respect to the other parameters of the problem and its bin-packing constant.

The models considered in this paper and in [1, 3–5, 7, 9, 15, 26] are based on some restrictive assumptions. First, it is assumed that the retailers can hold any quantity of stock where in practice this is not always the case. Second, the fleet size is assumed unbounded. In addition, travel times as well as loading and unloading times are not taken into account. Although these issues are important, we believe that they should be considered at a second phase in which the routes are assigned to vehicles/drivers and the specific time table is generated by using a scheduling algorithm. The capacity limits at the retailers, if binding, should be analyzed from an economical point of view: Are these capacities firm constraints or maybe it is worth investing in a capacity expansion at some of the retailers. For the evaluation of the benefit induced by a capacity expansion, one needs the information about the average cost while removing these restrictions which can be obtained by one of the methods described in the papers mentioned above. The issues of fleet size and the duration of the different tasks is related to the scheduling of the different tours which we do not consider at this level of the solution. In general, we can say that these methods produce solutions that have the potential to exploit the vehicles' at their maximum capacity depending on the effectiveness of the scheduling methods used in the second phase in order to coordinate the different routes.

These models further assume deterministic and constant demand rates. In our view this is the most unrealistic assumption. In today's world the technology allows for an easy accessibility to an accurate and timely information about the retailers stock levels. Unfortunately, the literature on IRP with stochastic demands confines itself to a single period models. One approach for solving multiperiod such problems with low variations in the demands is by using solution methods for the deterministic case where the mean of the demands plays the role of the demand rates. One of the most important criterions for determining the quality of an algorithm for a deterministic problem is its robustness to small variabilities in the demands. We expect that the algorithm proposed here is robust in the retailers' demand rates mainly when the problem does not allow perfect packing, since in such cases the vehicles leave the warehouse only partially loaded, enabling the flexibility of loading extra merchandise to fill unexpected increase in the demands. However, this scheme does not provide an answer with respect to the safety stocks needed at the retailers. We believe that one of the most challenging questions in IRP theory is the analysis of the multi-period stochastic IRP.

APPENDIX

PROOF OF LEMMA 7:

(a) At the end of Step 1 of CFP, in sectors $k = 1, \ldots, K - 1$ and possibly also in sector K, we have $|S_k| = \lfloor n^{5/8} \rfloor \lfloor n^{1/4} \rfloor \leq n^{7/8}$ retailers. In sector K we may have $n' \stackrel{\text{def}}{=} n - \lfloor n^{1/4} \rfloor \lfloor n^{5/8} \rfloor \lfloor n^{1/8} \rfloor$ retailers. A simple calculation shows that

$$n' \leq n^{7/8} + n^{3/4} - n^{5/8} + n^{3/8} - n^{1/4} - n^{1/8} + 1.$$

It is straightforward to show that $n^{7/8} \leq n^{7/8} + n^{3/4} - n^{5/8} + n^{3/8} - n^{1/4} - n^{1/8} + 1$ for all $n \geq 1$.

(b) We note that in Step 1 of CFP we add only radial cuts to the circle with radius ρ around the depot. Let $\tilde{\Pi}$ denote the perimeter of the clusters obtained at the end of Step 2 of CFP. We can bound $\tilde{\Pi}$ by bounding separately three of its components:

$$\Pi_1 = \text{the perimeter of the circle containing } N, \text{ i.e., } 2\pi\rho,$$

$$\Pi_2 = \text{the total length of all radial cuts generated in Step 1 of CFP},$$

$$\Pi_3 = \text{the total length of all circular cuts generated in Step 2 of CFP},$$

After Step 2 of CFP, the total perimeter $\tilde{\Pi}$ is thus bounded by $\tilde{\Pi} = \Pi_1 + 2(\Pi_2 + \Pi_3)$ since each circular or radial cut is adjacent to two clusters. Clearly $\Pi_1 = 2\pi\rho$ and $\Pi_2 = K\rho$. The number of circular cuts within sector k is bounded by:

$$\left\lceil \frac{|S_k|}{\lfloor n^{1/8} \rfloor} \right\rceil - 1 \leq \left\lceil \frac{n^{7/8} + n^{3/4} - n^{5/8} + n^{3/8} - n^{1/4} - n^{1/8} + 1}{\lfloor n^{1/8} \rfloor} \right\rceil - 1 \leq \frac{n^{7/8} + n^{3/4} - n^{5/8} + n^{3/8} - n^{1/4} - n^{1/8} + 1}{n^{1/8} - 1}$$
$$= n^{1/8}(n^{3/8} + 1)(n^{1/8} + 1)^2 + \frac{1}{n^{1/8} - 1}.$$

Observe now that $1/(n^{1/8} - 1) < 12$ for any $n \geq 2$. Thus the total length of all circular cuts in Step 2 is bounded from above by $n^{1/8}(n^{3/8} + 1)(n^{1/8} + 1)^2 + 12$ times the perimeter of the circle containing N, i.e., $\Pi_3 \leq 2\pi\rho(n^{1/8}(n^{3/8} + 1)(n^{1/8} + 1)^2 + 12)$. Thus,

$$\tilde{\Pi} = \Pi_1 + 2(\Pi_2 + \Pi_3) \leq 2\pi\rho + 2(K\rho + 2\pi\rho(n^{1/8}(n^{3/8} + 1)(n^{1/8} + 1)^2 + 12)) \leq 2\pi\rho + 2((\lfloor n^{1/4} \rfloor + 1)\rho$$
$$+ 2\pi\rho(n^{1/8}(n^{3/8} + 1)(n^{1/8} + 1)^2 + 12)).$$

Now (b) follows by simple algebra. □

PROOF OF LEMMA 10: From (3), we let

$$Z_j \stackrel{\text{def}}{=} \sum_{\ell \in \mathcal{C}_j'} \sum_{i \in \mathcal{B}_{j,\ell}} \bar{z}_\theta(2\|v_{j,\ell}\|, w(B_{j,\ell,i})), \qquad \text{for } j \in \{1, 2, 3\}.$$

Thus, (3) can be rewritten:

$$Z_n^{\mathrm{CFP(H)}} \le \sum_{j=1}^{3} Z_j + \mathcal{L}_n. \tag{10}$$

Let N_j (respectively, $\tilde{N}_j$) denote the set of retailers in A_j' (respectively, $\cup_\ell C_{j,\ell}$), for $j \in \{1, 2, 3\}$. Note that $N_1 \subseteq \tilde{N}_1$, $\tilde{N}_3 \subseteq N_3$, and $N_2 = (\tilde{N}_2 \backslash N_3) \cup (\tilde{N}_1 \backslash N_1)$. Define $n_j = |N_j|$ and $\tilde{n}_j = |\tilde{N}_j|$, for $j \in \{1, 2, 3\}$. We have $\tilde{n}_1 - n_1 \le K \cdot p = O(n^{3/8})$. Similar reasoning gives $n_3 - \tilde{n}_3 = O(n^{3/8})$ and $|n_2 - \tilde{n}_2| = O(n^{3/8})$. Then, for $j \in \{1, 2, 3\}$, $\lim_{n\to\infty} \tilde{n}_j/n = \lim_{n\to\infty} n_j/n = \mu(A_j')$.

Now in (10), we divide by n and take the limit. For this purpose, we define $\mathcal{J} \subseteq \{1, 2, 3\}$ such that $j \in \mathcal{J}$ if and only if $\mu(A_j') > 0$. Clearly, for $j \notin \mathcal{J}$, $\lim_{n\to\infty}(Z_j/n) = 0$ and for $j \in \mathcal{J}$, $n \to \infty$ implies $n_j \to \infty$ and also $\tilde{n}_j \to \infty$ almost surely. Using Lemma 8 results in (almost surely):

$$\overline{\lim_{n\to\infty}} \frac{1}{n} Z_n^{\mathrm{CFP(H)}} \le \sum_{j=1}^{3} \overline{\lim_{n\to\infty}} \frac{1}{n} Z_j = \sum_{j\in\mathcal{J}} \overline{\lim_{n\to\infty}} \frac{\tilde{n}_j}{n}\frac{Z_j}{\tilde{n}_j} \le \sum_{j\in\mathcal{J}} \left(\overline{\lim_{n\to\infty}} \frac{\tilde{n}_j}{n}\right)\left(\overline{\lim_{n\to\infty}} \frac{Z_j}{\tilde{n}_j}\right) = \sum_{j\in\mathcal{J}} \mu(A_j') \overline{\lim_{\tilde{n}_j\to\infty}} \frac{Z_j}{\tilde{n}_j}. \tag{11}$$

We consider the three rings A_1', A_2' and A_3' in turn. For each, we assume $\mu(A_j') > 0$; otherwise there is no need to bound the term for that ring since the respective expression vanishes. Recall that, for $j \in \{1, 2, 3\}$, $\mathcal{C}_j'$ is the index set of the clusters created by CFP whose closest point is in A_j'; i.e., if $\ell \in \mathcal{C}_j'$, then $v_{j,\ell} \in A_j'$. For any cluster $C_{j,\ell}$, let $d^{j,\ell} =^{\text{def}} \|v_{j,\ell}\|$. Recall the definition of the function $\bar{z}_\theta$ as given in Lemma 4. From here on, we set $\theta = \alpha^H$.

Consider ring A_1'. We have

$$Z_1 = \sum_{\ell\in\mathcal{C}_1'} \sum_{i\in\mathcal{B}_{1,\ell}} \bar{z}_\theta(2d^{1,\ell}, w(B_{1,\ell,i})) = \sum_{\ell\in\mathcal{C}_1'} \sum_{i\in\mathcal{B}_{1,\ell}} \left[\frac{hw(B_{1,\ell,i})}{(2f)} + (2d^{1,\ell} + c)f\right] = \sum_{i\in\tilde{N}_1} \frac{hw_i}{(2f)} + f \sum_{\ell\in\mathcal{C}_1'} b^H(C_{1,\ell})(2d^{1,\ell} + c).$$

Thus

$$\overline{\lim_{\tilde{n}_1\to\infty}} \frac{Z_1}{\tilde{n}_1} \le \frac{h}{(2f)} \overline{\lim_{\tilde{n}_1\to\infty}} \frac{1}{\tilde{n}_1} \sum_{i\in\tilde{N}_1} w_i + f\, \overline{\lim_{\tilde{n}_1\to\infty}} \frac{1}{\tilde{n}_1} \sum_{\ell\in\mathcal{C}_1'} b^H(C_{1,\ell})(2d^{1,\ell} + c) \le \frac{h\mathsf{E}[w]}{(2f)}$$

$$+ f\gamma^H \overline{\lim_{\tilde{n}_1\to\infty}} \frac{1}{\tilde{n}_1} \sum_{\ell\in\mathcal{C}_1'} (2d^{1,\ell} + c)|C_{1,\ell}| \le \frac{\beta hQ}{2} + f\gamma^H(2\bar{d}_1' + c). \tag{12}$$

The first inequality follows from the fact that H is robust-under-partitioning (see Property 9 with $\vartheta_\ell = 2d^{1,\ell} + c$ for $\ell \in \mathcal{C}_1'$). The last inequality follows from the law of large numbers and the fact that $\tilde{n}_1 = n_1 + o(n)$.

Now consider ring A_2':

$$Z_2 = \sum_{\ell\in\mathcal{C}_2'} \sum_{i\in\mathcal{B}_{2,\ell}} \bar{z}_\theta(2d^{2,\ell}, w(B_{2,\ell,i})) = \sum_{\ell\in\mathcal{C}_2'} \sum_{i\in\mathcal{B}_{2,\ell}} \left[w(B_{2,\ell,i}) \sqrt{\frac{(2d^{2,\ell} + c)h\alpha^H}{2Qf}} + \sqrt{\frac{(2d^{2,\ell} + c)hQf}{2\alpha^H}}\right]$$

$$\le \sum_{i\in\tilde{N}_2} w_i \sqrt{\frac{(2d_i + c)h\alpha^H}{2Qf}} + \sum_{\ell\in\mathcal{C}_2'} b^H(C_{2,\ell}) \sqrt{\frac{(2d^{2,\ell} + c)hQf}{2\alpha^H}}. \tag{13}$$

After dividing by $\tilde{n}_2$, taking the limit and noting that $|n_2 - \tilde{n}_2| = o(n)$, the first term of (13) becomes

$$\overline{\lim_{\tilde{n}_2\to\infty}} \frac{1}{\tilde{n}_2} \sum_{i\in\tilde{N}_2} w_i \sqrt{\frac{(2d_i + c)h\alpha^H}{2Qf}} = \sqrt{\frac{h\alpha^H}{2Qf}} \cdot \mathsf{E}[w] \cdot \mathsf{E}[\sqrt{2d + c} \mid 2d + c \in \Omega_2'] = \beta \sqrt{\frac{h\alpha^H Qf}{2}} \cdot \mathsf{E}[\sqrt{2d + c} \mid 2d + c \in \Omega_2']. \tag{14}$$

For the second term of (13), we use again the fact that H is robust-under-partitioning (see Property 9 with $\vartheta_\ell = \sqrt{2d^{2,\ell} + c}$ for $\ell \in \mathcal{C}_2'$) and get

$$\overline{\lim_{\tilde{n}_2\to\infty}}\frac{1}{\tilde{n}_2}\sum_{\ell\in\mathscr{C}_2'} b^H(C_{2,\ell})\sqrt{\frac{(2d^{2,\ell}+c)hQf}{2\alpha^H}} = \sqrt{\frac{hQf}{2\alpha^H}}\,\overline{\lim_{\tilde{n}_2\to\infty}}\frac{1}{\tilde{n}_2}\sum_{\ell\in\mathscr{C}_2'} b^H(C_{2,\ell})\sqrt{2d^{2,\ell}+c} \le \gamma^H\sqrt{\frac{hQf}{2\alpha^H}}\,\overline{\lim_{\tilde{n}_2\to\infty}}\frac{1}{\tilde{n}_2}\sum_{\ell\in\mathscr{C}_2'}|C_{2,\ell}|\sqrt{2d^{2,\ell}+c}$$

$$\le \gamma^H\sqrt{\frac{hQf}{2\alpha^H}}\cdot \mathsf{E}[\sqrt{2d+c}\mid 2d+c\in\Omega_2']. \quad (15)$$

This last inequality follows from the fact that $|n_2 - \tilde{n}_2| = o(n)$.

Now consider ring A_3'. We have

$$Z_3 = \sum_{\ell\in\mathscr{C}_3'}\sum_{i\in\mathscr{B}_{3,\ell}} \bar{z}_\theta(2d^{3,\ell}, w(B_{3,\ell,i})) = \sum_{\ell\in\mathscr{C}_3'}\sum_{i\in\mathscr{B}_{3,\ell}}\left[\frac{(2d^{3,\ell}+c)w(B_{3,\ell,i})}{Q}+\frac{hQ}{2}\right] \le \sum_{i\in\tilde{N}_3}\frac{(2d_i+c)w_i}{Q}+\left(\frac{hQ}{2}\right)\sum_{\ell\in\mathscr{C}_3'} b^H(C_{3,\ell}).$$

Thus, using again the fact that H is robust-under-partitioning (see Property 9 with $\vartheta_\ell = 1$ for $\ell \in \mathscr{C}_3'$), we get

$$\overline{\lim_{\tilde{n}_3\to\infty}}\frac{Z_3}{\tilde{n}_3} \le \overline{\lim_{\tilde{n}_3\to\infty}}\frac{1}{\tilde{n}_3}\sum_{i\in\tilde{N}_3}\frac{(2d_i+c)w_i}{Q}+\left(\frac{hQ}{2}\right)\overline{\lim_{\tilde{n}_3\to\infty}}\frac{1}{\tilde{n}_3}\sum_{\ell\in\mathscr{C}_3'} b^H(C_{3,\ell}) \le (2\bar{d}_3'+c)\beta f+\frac{hQ\gamma^H}{2}. \quad (16)$$

The first term arises since $n_3 - \tilde{n}_3 = o(n)$.

The result then follows by combining (12), (13), (14), (15), and (16) with (11). □

PROOF OF THEOREM 13: In what follows we have used the partitioning scheme with $\theta = \alpha^H$.

(a) According to Lemma 12, we need only determine the value of ξ^H. To simplify the presentation, we let $\nu \overset{\text{def}}{=} 2(2d + c)f/hQ$. In addition, recall that $\alpha = \gamma/\beta$ and $\alpha^H = \gamma^H/\beta$. Let $d \ge 0$.

Case 1: $2d + c \in \Omega_1 \cap \Omega_1' = \Omega_1 = [0, Qh/4f)$. This implies $0 \le \nu < \frac{1}{2}$ and thus

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} = \frac{\dfrac{hQ\beta}{2}+(2d+c)f\gamma^H}{\dfrac{hQ\beta}{2}+(2d+c)f\gamma} = \frac{\dfrac{hQ\beta}{2}+\dfrac{\gamma^H\nu hQ}{2}}{\dfrac{hQ\beta}{2}+\dfrac{\gamma\nu hQ}{2}} \le \frac{1+\dfrac{\alpha^H}{2}}{1+\dfrac{\alpha}{2}} \le \sqrt{r^H}.$$

This last inequality follows by using $r^H < 2$.

Case 2: $2d + c \in \Omega_3 \cap \Omega_3' = \Omega_3 = [Qh/f, \infty)$. This implies $\nu \ge 2$ and thus

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} = \frac{(2d+c)\beta f+\dfrac{\gamma^H hQ}{2}}{(2d+c)\beta f+\dfrac{\gamma hQ}{2}} = \frac{\dfrac{\nu hQ\beta}{2}+\dfrac{\gamma^H hQ}{2}}{\dfrac{\nu hQ\beta}{2}+\dfrac{\gamma hQ}{2}} \le \frac{1+\dfrac{\alpha^H}{2}}{1+\dfrac{\alpha}{2}} \le \sqrt{r^H}.$$

This last inequality follows by using $r^H < 2$.

We now consider the range $2d + c \in \Omega_2$. This is separated into three cases:

Case 3: $2d + c \in \Omega_2 \cap \Omega_1' = [Qh/4f, Qh/(2f\alpha^H))$ implies $\frac{1}{2} \le \nu < 1/\alpha^H$ and

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} = \frac{\dfrac{hQ\beta}{2}+(2d+c)f\gamma^H}{(2\beta-\gamma)\left((2d+c)f+\dfrac{hQ}{2}\right)+2(\gamma-\beta)\sqrt{(2d+c)hQf}} = \frac{1+\nu\gamma^H/\beta}{(2-\alpha)(\nu+1)+2\sqrt{2}(\alpha-1)\sqrt{\nu}}.$$

Taking the derivative of this ratio with respect to ν, we see that the sign of the derivative is the same as the sign of

$$(2-\alpha)\left(\frac{\gamma^H}{\beta}-1\right)+\sqrt{2}(\alpha-1)\left(\frac{\gamma^H}{\beta}\sqrt{\nu}-\frac{1}{\sqrt{\nu}}\right). \quad (17)$$

Note (17) is increasing in γ^H/β. Then, by using the facts that $\alpha < 2$ and $\gamma^H/\beta \geq \alpha \geq 1$ and assuming that γ^H/β is as low as α, we obtain that the sign of the derivative is the same as the sign of $[(\alpha - 1)/\sqrt{2\nu}][2\alpha\nu + (2 - \alpha)\sqrt{2\nu} - 2]$. This means that the derivative is nonnegative if $2\alpha\nu + (2 - \alpha)\sqrt{2\nu} - 2 \geq 0$. Investigation of this quadratic inequality in $\sqrt{\nu}$ reveals that the inequality holds for any $\nu \geq \frac{1}{2}$. Therefore the ratio is increasing in ν as long as $\nu \geq \frac{1}{2}$. To make the ratio as large as possible, we let $\nu = 1/\alpha^H$. The ratio can then be written as

$$\frac{2\alpha^H}{(2-\alpha)(\alpha^H + 1) + 2\sqrt{2}(\alpha - 1)\sqrt{\alpha^H}}. \tag{18}$$

This expression will be analyzed below when we compare bounds.

Case 4: $2d + c \in \Omega_2 \cap \Omega'_2$. Hence $2d + c \in [Qh/(2f\alpha^H), Qh\alpha^H/(2f)]$ which implies $1/\alpha^H \leq \nu \leq \alpha^H$. Simple algebra results in

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} = \frac{2\sqrt{\gamma^H\beta}}{(2\beta - \gamma)(\sqrt{\nu} + 1/\sqrt{\nu}) + 2\sqrt{2}(\gamma - \beta)}.$$

This ratio is clearly maximized at $\nu = 1$. By using $\alpha = \gamma/\beta$, we get that the maximum value for this ratio in this range is equal to

$$\frac{\sqrt{\alpha}}{(\sqrt{2} - 1)(\alpha + \sqrt{2})}\sqrt{r^H}. \tag{19}$$

Investigation of this expression as a function of α shows that it is maximized at $\alpha = \sqrt{2}$ and, therefore,

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} \leq \frac{\sqrt{\sqrt{2}}}{2\sqrt{2}(\sqrt{2} - 1)}\sqrt{r^H} \approx 1.015\sqrt{r^H}.$$

Case 5: $2d + c \in \Omega_2 \cap \Omega'_3$. Hence $2d + c \in [Qh\alpha^H/(2f), Qh/f)$ which implies $\alpha^H < \nu \leq 2$. Simple algebra shows that

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} = \frac{\nu\beta + \gamma^H}{(2\beta - \gamma)(\nu + 1) + 2(\gamma - \beta)\sqrt{2\nu}}.$$

Taking the derivative of this ratio with respect to ν shows that the sign of the derivative is the same as the sign of the expression $(2\beta - \gamma)(\beta - \gamma^H)\sqrt{\nu} + \sqrt{2}(\gamma - \beta)(\beta\nu - \gamma^H)$. Thus, the derivative is nonpositive if and only if $\sqrt{2}(\gamma - \beta)(\nu\beta - \gamma^H) \leq (2\beta - \gamma)(\gamma^H - \beta)\sqrt{\nu}$. We will show that, for any ν and $\alpha^H < \nu < 2$, this condition is satisfied. Indeed, we will show that, for any ν in this range, a stronger condition holds, namely, that $\sqrt{2}(\nu\beta - \gamma) \leq (2\beta - \gamma)\sqrt{\nu}$. Note that this last condition is stronger in view of the fact that $\beta \leq \gamma \leq \gamma^H$. We show this by demonstrating that the quadratic function (in $\sqrt{\nu}$) given by $\sqrt{2}\beta\nu - (2\beta - \gamma)\sqrt{\nu} - \sqrt{2}\gamma$ is nonpositive. The function's only positive root is $\sqrt{2}$, proving that the quadratic function is nonpositive for any ν in the range $0 \leq \nu \leq 2$ and, in particular, for any $\alpha^H < \nu \leq 2$. As a result, we obtain that the ratio is decreasing in ν in the relevant range and therefore its maximum is obtained at $\nu = \alpha^H$. A simple calculation shows that at $\nu = \alpha^H$

$$\frac{\bar{c}^H(d)}{\underline{c}(d)} \leq \frac{2\alpha^H}{(2-\alpha)(\alpha^H + 1) + 2\sqrt{2}(\alpha - 1)\sqrt{\alpha^H}}.$$

This is exactly (18), i.e., the same as in Case 3.

It is possible to show (using simple algebra) that the bound in Cases 3 and 5 is dominated by the one in Case 4 [i.e., the expression in (18) is no more than (19)]. The bound in Cases 1 and 2 is also dominated by (19). Therefore, we obtain that CFP(H) is an asymptotic $1.015\sqrt{r^H}$-approximation algorithm for the best FPP. (b) follows directly from (a) by substituting $r^H = 1$. □

The following technical Lemma is needed in the proof of Lemma B.

LEMMA 20:

(a) Let Y_k, for $k = 1, 2, \ldots, K$, be independent and discrete random variables. Then, for any $v \in \mathbb{R}$, $t > 0$, and any two vectors of constants $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_K)$:

$$\mathsf{P}\left[\sum_{k=1}^{K} \alpha_k \cdot |Y_k - \lambda_k| \geq v\right] \leq e^{-vt} \cdot \prod_{k=1}^{K} \mathsf{E}(e^{t\alpha_k \cdot |Y_k - \lambda_k|}).$$

(b) If, in addition, Y_k is a Poisson(λ_k) random variable, for $k = 1, \ldots, K$ and $0 < \alpha_k \leq \bar{\alpha}$ for $k = 1, \ldots, K$, then

$$\mathsf{P}\left[\sum_{k=1}^{K} \alpha_k \cdot |Y_k - \lambda_k| \geq v\right] \leq 2^K \cdot e^{\Lambda(e^{\bar{\alpha}t} - 1 - \bar{\alpha}t) - vt},$$

where $\Lambda \stackrel{\text{def}}{=} \sum_{k=1}^{K} \lambda_k$.

PROOF: We first show that if Y is a discrete random variable, then for any $v \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $t > 0$,

$$\mathsf{P}[\alpha \cdot |Y - \lambda| \geq v] \leq \mathsf{E}(e^{t(\alpha \cdot |Y - \lambda| - v)}).$$

To see this, let $\mathbf{1}_{(A)}$ denote the indicator of A (1 if A is true, 0 otherwise) and let $p(k) = \mathsf{P}(Y = k)$:

$$\mathsf{P}(\alpha \cdot |Y - \lambda| \geq v) = \sum_{\{k\,:\,\alpha \cdot |k - \lambda| \geq v\}} p(k) = \sum_{k=-\infty}^{\infty} \mathbf{1}_{(\alpha \cdot |k - \lambda| \geq v)} p(k) \leq \sum_{k=-\infty}^{\infty} e^{t(\alpha \cdot |k - \lambda| - v)} p(k) = \mathsf{E}[e^{t(\alpha \cdot |k - \lambda| - v)}]. \quad (20)$$

The inequality follows since for all $t > 0$, $e^{t(\alpha|k-\lambda|-v)} \geq 0$ and if $\alpha \cdot |k - \lambda| \geq v$ then $e^{t(\alpha \cdot |k - \lambda| - v)} \geq 1$.

Now suppose that, for $k = 1, \ldots, K$, Y_k are independent discrete random variables with $p_i(m) = \mathsf{P}[Y_i = m]$. Given two vectors of constants $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_K)$, let $v \in \mathbb{R}$ and let M_v denote all vectors $z = (z_1, z_2, \ldots, z_K)$ of the infinite K-dimensional integer grid that satisfy $\sum_{k=1}^{K} \alpha_k \cdot |z_k - \lambda_k| \geq v$. For $t > 0$,

$$\mathsf{P}\left[\sum_{k=1}^{K} \alpha_k \cdot |Y_k - \lambda_k| \geq v\right] = \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} \cdots \sum_{\ell_K=-\infty}^{\infty} \mathbf{1}_{\{(\ell_1, \ell_2, \ldots, \ell_K) \in M_v\}} p_1(\ell_1) p_2(\ell_2) \cdots p_K(\ell_K)$$

$$\leq \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} \cdots \sum_{\ell_K=-\infty}^{\infty} e^{t(\sum_{k=1}^{K} \alpha_k \cdot |\ell_k - \lambda_k| - v)} \prod_{k=1}^{K} p_k(\ell_k) \qquad = e^{-vt} \cdot \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2=-\infty}^{\infty} \cdots \sum_{\ell_K=-\infty}^{\infty} \prod_{k=1}^{K} e^{t\alpha_k |\ell_k - \lambda_k|} p_k(\ell_k)$$

$$= e^{-vt} \cdot \prod_{k=1}^{K} \sum_{\ell_k=-\infty}^{\infty} e^{t\alpha_k |\ell_k - \lambda_k|} p_k(\ell_k) = e^{-vt} \cdot \prod_{k=1}^{K} \mathsf{E}[e^{t\alpha_k |Y_k - \lambda_k|}]. \quad \text{(see (20))}$$

This proves part (a).

It remains to prove (b). Assume Y_k is Poisson(λ_k) for $k = 1, \ldots, K$ and $0 < \alpha_k \leq \bar{\alpha}$. Note that if Y is a Poisson(λ) random variable, then, for any $\alpha > 0$,

$$\mathsf{E}[e^{t\alpha|Y-\lambda|}] \leq \mathsf{E}[e^{t\alpha(Y-\lambda)} + e^{t\alpha(\lambda - Y)}] = \mathsf{E}[e^{\alpha t Y - \alpha t \lambda} + e^{\alpha t \lambda - \alpha t Y}] = e^{\lambda(e^{\alpha t} - 1)} \cdot e^{-t\lambda\alpha} + e^{\lambda(e^{-t\alpha} - 1)} \cdot e^{t\lambda\alpha}$$

$$\leq 2e^{\lambda(e^{\alpha t} - 1 - \alpha t)} \qquad (\text{since } e^{-u} + u \leq e^{u} - u, \quad \forall\, u > 0.) \quad (21)$$

The second equality follows from the moment generating function of a Poisson random variable, that is $\mathsf{E}[e^{uY}] = e^{\lambda(e^u - 1)}$.

Let $\Lambda \stackrel{\text{def}}{=} \sum_{k=1}^{K} \lambda_k$. Now combining (21) and part (a), we get

$$\mathsf{P}\left[\sum_{k=1}^{K} \alpha_k \cdot |Y_k - \lambda_k| \geq v\right] \leq e^{-vt} \cdot \prod_{k=1}^{K} \mathsf{E}(e^{t\alpha_k \cdot |Y_k - \lambda_k|}) \leq e^{-vt} \cdot \prod_{k=1}^{K} (2e^{\lambda_k(e^{\alpha_k t} - 1 - \alpha_k t)}) \leq 2^K \cdot e^{\Lambda(e^{\bar{\alpha}t} - 1 - \bar{\alpha}t) - vt}. \qquad \square$$

LEMMA 21: If $a(n) = o(n)$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell |\bar{n}(C_\ell) - \mathsf{E}[\bar{n}(C_\ell)]| = 0 \qquad \text{(a.s.)}.$$

PROOF: For each $\ell = 1, 2, \ldots, a(n)$ let $n_\ell = |C_\ell|$. For the proof, we use a Poissonization (as in [13]). Consider a cluster $\ell \in \{1, 2, \ldots, a(n)\}$. Let $\xi^\ell = \{\xi_1^\ell, \xi_2^\ell, \ldots\}$ be a sequence of independent $\{1, 2, \ldots, I\}$-valued random variables distributed according to $\mathsf{P}[\xi_j^\ell = i] = p_i \overset{\text{def}}{=} \mathsf{P}(w \in \mathcal{I}_i)$ for $1 \le i \le I$ and $j \ge 1$. Let $X_{(\ell,i)}^n$ be the number of occurrences of the value i among $\xi_1^\ell, \xi_2^\ell, \ldots, \xi_{n_\ell}^\ell$, and thus $X_{(\ell,i)}^n$ is distributed as $\bar{n}_i(C_\ell)$, for all i, ℓ. Let N_ℓ be a Poisson(n_ℓ) random variable independent of ξ^ℓ, and let $Y_{(\ell,i)}^n$ be the number of occurrences of the value i among $\xi_1^\ell, \xi_2^\ell, \ldots, \xi_{N_\ell}^\ell$. Clearly, $Y_{(\ell,1)}^n, Y_{(\ell,2)}^n, \ldots, Y_{(\ell,I)}^n$ are independent Poisson random variables with means $n_\ell p_1, n_\ell p_2, \ldots, n_\ell p_I$, and that $X_{(\ell,1)}^n, X_{(\ell,2)}^n, \ldots, X_{(\ell,I)}^n$ is a multinomial $(n_\ell, p_1, p_2, \ldots, p_I)$ random vector.

Define, for all $n \ge 1$,

$$U_n \overset{\text{def}}{=} \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |X_{(\ell,i)}^n - \mathsf{E}[X_{(\ell,i)}^n]| \ge 0.$$

Our goal is to show that $\lim_{n\to\infty} U_n = 0$, almost surely. First note that $\mathsf{E}[X_{(\ell,i)}^n] = n_\ell p_i$, for all i, ℓ. We have (for all i, ℓ) $|X_{(\ell,i)}^n - n_\ell p_i| \le |X_{(\ell,i)}^n - Y_{(\ell,i)}^n| + |Y_{(\ell,i)}^n - n_\ell p_i|$. Hence

$$U_n \le \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |X_{(\ell,i)}^n - Y_{(\ell,i)}^n| + \frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |Y_{(\ell,i)}^n - n_\ell p_i|. \tag{22}$$

Now pick $\epsilon_0 \in (0, 1)$ such that $\epsilon_0 \le 2(2\rho + c)$. Then, from (22),

$$\mathsf{P}[U_n > \epsilon_0] \le \mathsf{P}\left[\frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |X_{(\ell,i)}^n - Y_{(\ell,i)}^n| > \frac{\epsilon_0}{2}\right] + \mathsf{P}\left[\frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |Y_{(\ell,i)}^n - n_\ell p_i| > \frac{\epsilon_0}{2}\right]. \tag{23}$$

We start by considering the first term on the right-hand side of (23). For each $\ell = 1, 2, \ldots, a(n)$, note that $\sum_{i=1}^{I} |X_{(\ell,i)}^n - Y_{(\ell,i)}^n| = |N_\ell - n_\ell|$. So

$$\mathsf{P}\left[\frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |X_{(\ell,i)}^n - Y_{(\ell,i)}^n| > \frac{\epsilon_0}{2}\right] = \mathsf{P}\left[\frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell |N_\ell - n_\ell| > \frac{\epsilon_0}{2}\right]$$

$$= \mathsf{P}\left[\sum_{\ell=1}^{a(n)} \vartheta_\ell |N_\ell - n_\ell| > \frac{n\epsilon_0}{2}\right] \le 2^{a(n)} \cdot e^{n(e^{\bar\vartheta t} - 1 - \bar\vartheta t) - nt\epsilon_0/2}, \tag{24}$$

for fixed $t > 0$. The last inequality holds as follows from Lemma 20, part (b), and $\sum_{\ell=1}^{a(n)} n_\ell = n$.

Now consider the second term of the right-hand side of (23). Here

$$\mathsf{P}\left[\frac{1}{n} \sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |Y_{(\ell,i)}^n - n_\ell p_i| > \frac{\epsilon_0}{2}\right] = \mathsf{P}\left[\sum_{\ell=1}^{a(n)} \vartheta_\ell \sum_{i=1}^{I} |Y_{(\ell,i)}^n - n_\ell p_i| > \frac{n\epsilon_0}{2}\right] \le 2^{a(n)I} \cdot e^{n(e^{\bar\vartheta t} - 1 - \bar\vartheta t) - nt\epsilon_0/2} \tag{25}$$

for fixed $t > 0$. This last inequality follows from Lemma 20, part (b), and $\sum_{\ell=1}^{a(n)} \sum_{i=1}^{I} n_\ell p_i = n$.

Finally combining (23) with (24) and (25):

$$\mathsf{P}[U_n > \epsilon_0] \le 2^{a(n)} \cdot e^{n(e^{\bar\vartheta t} - 1 - \bar\vartheta t) - nt\epsilon_0/2} + 2^{a(n)I} \cdot e^{n(e^{\bar\vartheta t} - 1 - \bar\vartheta t) - nt\epsilon_0/2} \le 2^{a(n)I+1} \cdot e^{n(e^{\bar\vartheta t} - 1 - \bar\vartheta t) - nt\epsilon_0/2}.$$

Now let $t = \bar\vartheta^{-1} \ln(1 + \epsilon_0/(2\bar\vartheta))$ we get

$$\mathsf{P}[U_n > \epsilon_0] \le 2^{a(n)I+1} \cdot e^{n[\epsilon_0/(2\bar\vartheta) - \ln(1+\epsilon_0/(2\bar\vartheta))] - n\epsilon_0 \ln(1+\epsilon_0/(2\bar\vartheta))/(2\bar\vartheta)} = 2^{a(n)I+1} \cdot e^{n[\epsilon_0/(2\bar\vartheta) - (1+\epsilon_0/(2\bar\vartheta))\ln(1+\epsilon_0/(2\bar\vartheta))]}.$$

Now using the fact that, $\forall x \in [0, 1]$, $x - (1 + x)\ln(1 + x) \le -x^2/[2(1 + x)]$, we get

$$\mathsf{P}[U_n > \epsilon_0] \le 2^{a(n)I+1} \cdot e^{-n(\epsilon_0/(2\bar\vartheta))^2/[2(1+\epsilon_0/(2\bar\vartheta))]} \le 2^{a(n)I+1} \cdot e^{-n\epsilon_0^2/[4\bar\vartheta(2\bar\vartheta+1)]} \qquad (\text{since } \epsilon_0 < 1)$$

$$\le e^{a(n)I+1-n\epsilon_0^2/[4\bar\vartheta(2\bar\vartheta+1)]} \qquad (\text{since } e > 2).$$

Now let n_0 be the smallest integer that satisfies $n_0 \geq [8\vartheta(2\vartheta + 1)] \cdot [a(n_0)I + 1]/\epsilon_0^2$. Then, $\forall n \geq n_0$, we have $\mathsf{P}[U_n > \epsilon_0] \leq e^{-n\epsilon_0^2/[8\bar{\vartheta}(2\bar{\vartheta}+1)]}$. Thus, $\sum_n \mathsf{P}(U_n > \epsilon_0) \leq \sum_n e^{-n\epsilon_0^2/[8\bar{\vartheta}(2\bar{\vartheta}+1)]} < +\infty$. Now invoking the Borel-Cantelli Lemma (see, e.g., [23]), this implies that $\overline{\lim}_{n\to\infty} U_n \leq 0$ almost surely. Since $U_n \geq 0$, this means $\lim_{n\to\infty} U_n = 0$ almost surely. □

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